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THE STRUCTURE OF THE SET OF SINGULAR POINTS OF A CODIMENSION 1 DIFFERENTIAL SYSTEM ON A 5-MANIFOLD

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ABSTRACT. Generic modules V of vector fields tangent to a 5-dimensional smooth manifold M, generated locally by four not necessarily linearly independent fields X_1, X_2, X_3, X_4, are considered. Denoting by ω the 1-form $X_4 \lrcorner X_3 \lrcorner X_2 \lrcorner X_1 \lrcorner \overset{5}{\Omega}$ conjugated to V ($\overset{5}{\Omega}$ is a fixed local volume form on M), the loci of singular behavior of V: $M_{\text{deg}}(V) = \{p \in M | \omega(p) = 0\}$ and $M_{\text{sing}}(V) = \{p \in M | \omega \wedge (d\omega)^2(p) = 0\}$ are handled. The local classification of this pair of sets is carried out (outside a curve and a discrete set in $M_{\text{deg}}(V)$) up to a smooth diffeomorphism. In the most complicated case, around points of a codimension 3 submanifold of M, $M_{\text{sing}}(V)$ turns out to be diffeomorphic to the Cartesian product of $\mathbb{R}^2$ and the Whitney's umbrella in $\mathbb{R}^3$.

1

We are going to consider generic differential systems of codimension 1 in the tangent bundle over a C^∞ manifold M of dimension 5. Unfortunately, this notion in different papers is used in many senses. We mean by it the module generated locally, over the ring of smooth functions on M, by four vector fields (the fields may happen to be linearly dependent at some points).

The equivalent framework for this investigation is that of singularities of k-tuples of vector fields on $\mathbb{R}^n$ set in [JP] (here $k = 4$, $n = 5$). The paper can be considered as a continuation of similar research in dimension 3 (included primarily in [JP]) and in dimension 4 [MR, M1–M3]. The study will be local, so that we shall often use the language of germs of sets, functions, vector fields, etc.

2

We shall assume once and for all that all the considered objects are C^∞ smooth, and this will not be additionally inserted in the statements. A point $p \in M$ is of interest to us when the germ at p of the considered system V is not equivalent to the Darboux model

$$\operatorname{span}\left(\frac{\partial}{\partial y} - u\frac{\partial}{\partial x}, \frac{\partial}{\partial z} - v\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right),$$

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or, equivalently, to the Pfaffian equation $dx + u\,dy + v\,dz = 0$ (throughout this paper we use x, y, z, u, v rather than x_1, x_2, y_1, y_2, z).

In other words, if (all this locally) Ω is a volume form on M and X_1, X_2, X_3, X_4 generate V, then one can define a 1-form conjugated to V, $\omega(\cdot) := \Omega(X_1, X_2, X_3, X_4, \cdot)$, which vanishes at every point p where $\dim V(p) \leq 3$. Such ω is not defined invariantly, but the Pfaffian equation it represents already is. So we are interested in either (a) $\omega(p) = 0$ or (b) $\omega(p) \neq 0$ and $\omega \wedge (d\omega)^2|_p = 0$. The latter means that the class of the Pfaffian equation $\omega = 0$ at p is not 5 (regarding this notion, see [F, Ma]).

Throughout this paper the union of the geometric loci of (a) and (b) is denoted by $M_{\text{sing}}(V)$, and the locus of (a) alone by $M_{\text{deg}}(V)$.

Note. We consider only typical degenerations, i.e., those that are unavoidable under arbitrary small perturbations of V. Therefore we assume that $\dim V(p) = 3$ for every $p \in M_{\text{deg}}(V)$ (the falling of $\dim V(p)$ by 2 is already a codimension 6 feature and is not typical by Transversality theorem, see [AGV, Ma]).

3

The normal forms of the smooth classification of germs of generic V's were found at generic points of $M_{\text{sing}}(V)\backslash M_{\text{deg}}(V)$ (i.e., in codimension 1) by Martinet [Ma], and recently at points of certain 2-dimensional surface included in it (where, typically, the next degeneration having codimension 3 materializes) by Zhitomirskii [Z1, Z2]. All these normal forms are simple, i.e., moduleless; such normal forms are called local models.

As regards $M_{\text{deg}}(V)$, which typically has codimension 2 (see Proposition in §7), even at its generic points normal forms are unknown. This is in contradistinction to dimensions 3 and 4, where local models at such points of the respective loci of $\dim V$ falling by 1 were found by Jakubczyk and Przytycki [JP] in the case of $\dim M = 3$, and by Mormul and Roussarie [MR] for $\dim M = 4$.

We suspect that in dimension 5 at points of $M_{\text{deg}}(V)$ there are no models, and moreover normal forms contain functional parameters. Yet already the problem of classification of the pair of germs of sets $M_{\text{sing}}(V)$, $M_{\text{deg}}(V)$ turns out to be interesting. The paper is devoted to this problem.

4

Main Theorem. *For a generic differential system V on M there exist subsets of $M_{\text{deg}}(V)$: a curve M_1 and a set of isolated points M_2 such that the germ of $(M_{\text{sing}}(V), M_{\text{deg}}(V))$ at any point of $M_{\text{sing}}(V)\backslash(M_1 \cup M_2)$ is equivalent to one of the following germs*:

(A) *germ of* 4-*manifold,* $\varnothing$;

(B) *germ of* 3-*manifold,* $M_{\text{deg}}(V) = M_{\text{sing}}(V)$;

(C) *germ of stratified manifold with strata of dimensions* 4, 4, 3 (*the last in the intersection of closures of the first and second*), $M_{\text{deg}}(V) =$ *the* 3-*dimensional stratum* (*see Figure* 1);

(D) *the germ of the Whitney's umbrella* $\times\mathbb{R}^2$, (*its handle*) $\times\mathbb{R}^2$ (*see Figure* 2).

In other words, in suitable coordinates x, y, z, u, v, the pair of sets $M_{\text{sing}}(V)$, $M_{\text{deg}}(V)$ is locally given by the equations

(A) $M_{\text{sing}} = \{x = 0\}$, $M_{\text{deg}} = \varnothing$;

(B) $M_{\text{sing}} = M_{\text{deg}} = \{x = y = 0\}$;

(C) $M_{\text{sing}} = \{xy = 0\}$, $M_{\text{deg}} = \{x = y = 0\}$;

(D) $M_{\text{sing}} = \{x^2 = zy^2\}$, $M_{\text{deg}} = \{x = y = 0\}$.

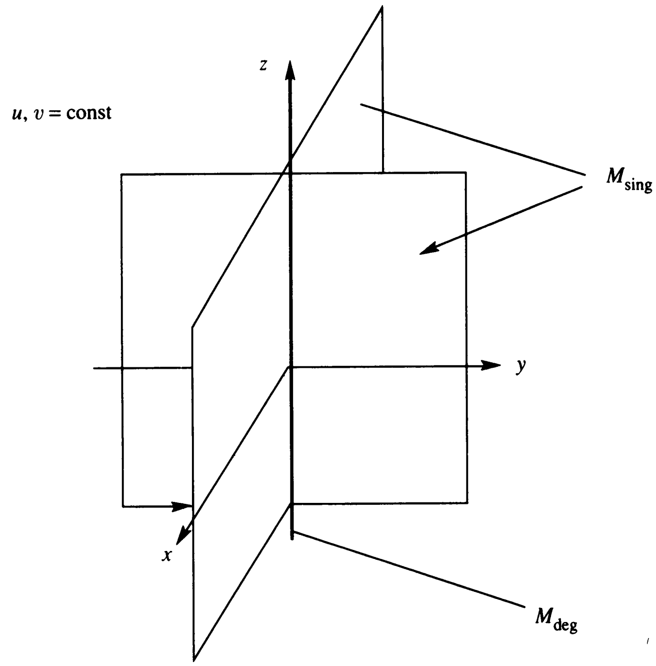

FIGURE 1

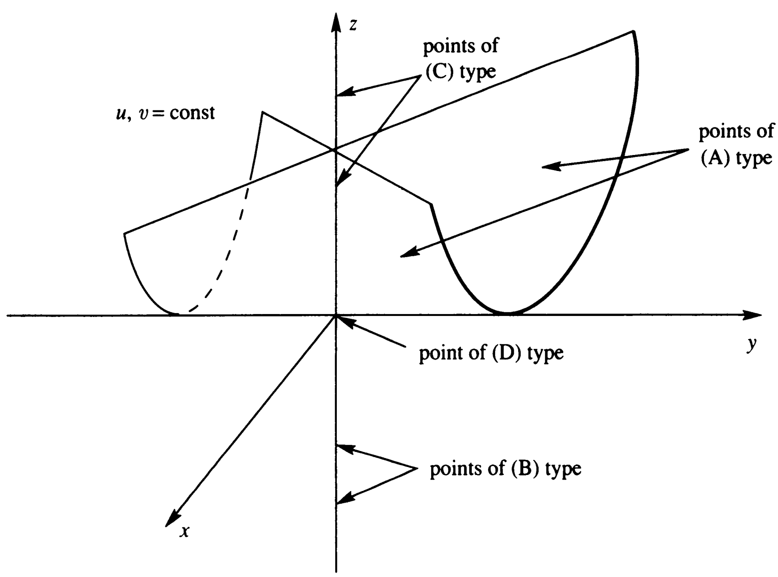

FIGURE 2

In cases (B)–(D), *in the mentioned coordinates,*

$$V(0) = \operatorname{span}(\partial/\partial x, \partial\partial/\partial y, \partial/\partial z).$$

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As we said in §3, in cases (B)–(D) local models probably do not exist. Nevertheless, we are able then to simplify (locally) V significantly. This, among other things, will come in handy in proving the Main Theorem in §11.

Let $\mathfrak{m}_{x,y}$ stand for the ideal of germs at $0 \in \mathbb{R}^5$ of functions vanishing on $\{x = y = 0\}$; $\mathfrak{m}_{x,y}^k$ is its kth power. For brevity we write the same symbol for the set of germs at 0 of 1-forms with coefficients in $\mathfrak{m}_{x,y}^k$. In the sequel $j_{x,y}^k$ will denote the natural projection $\mathscr{F}_0^5 \to \mathscr{F}_0^5/\mathfrak{m}_{x,y}^{k+1}$ ($\mathscr{F}_0^5 =$ the ring of germs at 0 of smooth functions on $\mathbb{R}^5$). We shall apply $j_{x,y}^k$ to germs of 1-forms in the natural sense, too.

The mentioned simplified description of V (a normal form) is given in terms of the conjugated form ω (cf. §2).

Theorem on normal form. *Let $p \in M_{\deg}(V)\backslash M_1$. There exist coordinates x, y, z, u, v (vanishing at p) s.t. the germ of ω at p has the form*

$$x\,du + y\,dv + f\,dz, \qquad f \in \mathfrak{m}_{x,y}^2. \tag{1}$$

Observe that in these coordinates

$$M_{\deg}(V) = \{x = y = 0\}. \tag{2}$$

Corollary 1. *At any $p \in M_{\deg}(V)$, $j_p^1(\omega \wedge (d\omega)^2) = 0$.*

Indeed, in the normal form coordinates

$$\omega \wedge (d\omega)^2 = 2(xf_x + yf_y - f)\,dx \wedge dy \wedge dz \wedge du \wedge dv, \tag{3}$$

and $f, xf_x, yf_y \in \mathfrak{m}_{x,y}^2$. (Here and in the sequel the symbol of a function followed by a lowercase letter subscript denotes the respective function's partial derivative.) □

Remark 1. In the normal form coordinates Corollary 1 can be written compactly as $j_{x,y}^1(\omega \wedge (d\omega)^2) = 0$.

Corollary 2. *Let $p \in M_{\deg}(V)\backslash M_1$. There exist coordinates x, y, z, u, v such that the germ of V at p is generated by vector fields $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z + f_1\partial/\partial u + f_2\partial/\partial v$, $y\partial/\partial u - x\partial/\partial v$, where $f_1, f_2 \in \mathfrak{m}_{x,y}$.*

Proof. Let ω be the 1-form conjugated to V and x, y, z, u, v be the coordinates of n.f. (1). We can write $f = -xf_1 - yf_2$ with $f_1, f_2 \in \mathfrak{m}_{x,y}$. Any v.f. ξ from V,

$$\xi = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z} + A_4\frac{\partial}{\partial u} + A_5\frac{\partial}{\partial v},$$

satisfies $\omega(\xi) \equiv 0$ or, equivalently, $xA_4 + yA_5 - (xf_1 + yf_2)A_3 \equiv 0$. The latter relation implies $A_4 - f_1A_3 = yg$ and $A_5 - f_2A_3 = -xg$ for some function g. Thus

$$\xi = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\left(\frac{\partial}{\partial z} + f_1\frac{\partial}{\partial u} + f_2\frac{\partial}{\partial v}\right) + g\left(y\frac{\partial}{\partial u} - x\frac{\partial}{\partial v}\right). \quad \square$$

Remark 2. It follows from Corollary 2 that the generators of V and their first order Lie brackets span at 0 the full 5-dimensional tangent space. (This property of V is weaker than the transversality assumed in §8 and occurs also without that transversality.)

We postpone the proof of the theorem on normal form till §12; the final part of it occupies §13.

6

In proving the Main Theorem we shall represent V by a triple (ω_1, ω_2, Y), where ω_1, ω_2 are 1-forms, and Y is a vector field.

Observation 1. *For $p \in M_{\deg}(V)$ there exist germs at p of a vector field Y and of independent Pfaffian forms ω_1 and ω_2 such that*

$$\omega := Y \lrcorner (\omega_1 \wedge \omega_2) \tag{4}$$

is conjugated to V and $M_{\deg}(V) = \{\omega_1(Y) = \omega_2(Y) = 0\}$.

Passing to the (ω_1, ω_2, Y)'s is purposeful, since any ω conjugated to V is highly nontypical among all differential 1-forms (it vanishes on a "big" set $M_{\deg}(V)$), while the objects in the triple (ω_1, ω_2, Y) do not vanish at any point.

Proof. Let V be generated by four vector fields X_1, X_2, X_3, Y, three of which are independent (for instance, X_1, X_2, X_3). The distribution spanned by them can be described by a Pfaffian system $\omega_1 = \omega_2 = 0$, and then $V = \operatorname{span}(Y, \ker\omega_1 \cap \ker\omega_2)$. Now the statements of the observation are easy to verify. □

It seems to us that the above representation of ω is of its own interest and can be applied in many situations.

7

Our first step in the proof of the Main Theorem consists in ensuring that

Proposition. *For V generic $M_{\deg}(V)$ is, if not empty, a smooth codimension 2 submanifold of M.*

Proof. We use description (4) and consider the set Q_1 of 1-jets at $0 \in \mathbb{R}^5$ of 3-tuples (ω_1, ω_2, Y). Let its subset $\widetilde{Q}_1$ be given by equations

$$\omega_1(Y)(0) = \omega_2(Y)(0) = 0, \qquad d(\omega_1(Y)) \wedge d(\omega_2(Y))|_0 = 0.$$

Clearly $\widetilde{Q}_1$ has codimension 6 in Q_1. The standard use of Transversality theorem gives that for generic V its 1-jet is nowhere included in $\widetilde{Q}_1$. This implies the conclusion of the Proposition. □

Remark 3. The theory mentioned in §1, developed in [JP], yields, among many other things, that the locus of $\dim V = k - 1$ is smooth for a generic codimension $n - k$ differential system in $\mathbb{R}^n$.

8

Here is the definition (in invariant terms) of the sets $M_1, M_2 \subset M_{\deg}(V)$ occurring in the Main Theorem.

A point p is included in M_1 iff $V(p)$ is not transversal to $M_{\deg}(V)$; a point p is included in M_2 iff $j_p^2(\omega \wedge (d\omega)^2) = 0$.

9

Now we are going to prove

(i) M_1 *is a smooth curve.*

The proof is based on the following

Lemma 1. *For a generic system* V, *at any* $p \in M_{\text{deg}}(V)$, $V(p)$ *is transversal to* $M_{\text{deg}}(V) \Leftrightarrow \operatorname{rank}(d\omega)^2(p) = 4$.

Proof. We take the ω from Observation 1. Computing at a point of $M_{\text{deg}}(V)$ and using Observation 1, we get $d\omega = d(\omega_1(Y)) \wedge \omega_2 - d(\omega_2(Y)) \wedge \omega_1$. Thus $(d\omega)^2 = -2\omega_1 \wedge \omega_2 \wedge d(\omega_1(Y)) \wedge d(\omega_2(Y))$. The condition $\operatorname{rank}(d\omega)^2 = 4$ means that the kernels of the 1-forms entering the above formula intersect one another as sparingly as possible.

Because for $p \in M_{\text{deg}}(V)$, $V(p) = \ker \omega_1(p) \cap \ker \omega_2(p)$, and since, in view of $d(\omega_1(Y)) \wedge d(\omega_2(Y)) \neq 0$, $M_{\text{deg}}(V)$ is a smooth manifold and $T_p M_{\text{deg}}(V) = \ker d(\omega_1(Y)) \cap \ker d(\omega_2(Y))$ (cf. Proposition in §7 and Observation 1), we conclude that so intersect each other $V(p)$ and $T_p M_{\text{deg}}(V)$, which means transversality.

The implication $\Rightarrow$ uses the same arguments and the condition of smoothness of $M_{\text{deg}}(V)$ valid for a generic system V. □

Note. The statement on the right-hand side of Lemma 1 is here equivalent to $(d\omega)^2(p) \neq 0$.

Proof of (i). Let $p \in M_1$. By Lemma 1 $\operatorname{rank}(d\omega)^2(p) < 4$, which means that the 1-jet at p of (ω_1, ω_2, Y) satisfies the conditions

$$\omega_1(Y)(p) = 0, \quad \omega_2(Y)(p) = 0, \quad \omega_1 \wedge \omega_2 \wedge d(\omega_1(Y)) \wedge d(\omega_2(Y))|_p = 0.$$

It is clear that these conditions distinguish a (stratified) codimension 4 manifold in the space Q_1 of 1-jets. One can show that its singular points form a set of codimension 6 in Q_1. Therefore (i) follows from Transversality theorem. □

10

In turn, we can show that

(ii) M_2 *consists of isolated points.*

Proof. M_2 is defined invariantly (see §8), and we prefer to work in the coordinates giving (1). Let

$$(5) \quad f = A(z, u, v)x^2 + B(z, u, v)xy + C(z, u, v)y^2 + \tilde{f}, \qquad \tilde{f} \in \mathfrak{m}^3_{x,y}.$$

Then in view of (2) and (3) M_2 is given locally by the equations

$$x = y = A = B = C = 0.$$

The application of Transversality theorem completes the proof. □

11. Proof of the Main Theorem

(A) This is the well-known case: for V generic in the sense of being transversal (after natural identifications) to the stratification C of 1-jets of the Pfaffian equations on M, constructed in [Ma, p. 136], $M_{\text{sing}}(V) \backslash M_{\text{deg}}(V)$ is a smooth codimension 1 submanifold of M.

Now consider the case $p \in M_{\deg}(V)\backslash(M_1 \cup M_2)$. We are using the normal form (1) coordinates. Putting $g := xf_x + yf_y - f$ and using (5), one has

$$g = A(z, u, v)x^2 + B(z, u, v)xy + C(z, u, v)y^2 + \hat{f}, \qquad \hat{f} \in \mathfrak{m}^3_{x,y}.$$

By (3), $(M_{\text{sing}}(V), M_{\deg}(V)) = (\{g = 0\}, \{x = y = 0\})$. Now to prove the Main Theorem it suffices to prove that there exists a local diffeomorphism Φ preserving the manifold $\{x = y = 0\}$ and such that $g \circ \Phi = \mp(x^2 + y^2)$, or $g \circ \Phi = xy$, or $g \circ \Phi = x^2 - y^2 z$. As we are outside M_2, $j^2_p g \neq 0$. We can assume $p = 0 \in \mathbb{R}^5$. Suppose at first that

$$\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix}(0) \neq 0. \tag{6}$$

Two subcases are possible:

(B) the Hessian is positively or negatively defined, and

(C) (6) holds and the Hessian is neither positively nor negatively defined.

By the Morse lemma with parameters [AGV] we can claim the existence of a coordinate change

$$x \to x + \phi(x, y, z), \quad y \to y + \psi(x, y, z), \qquad \phi, \psi \in \mathfrak{m}^3_{x,y}, \tag{7}$$

simplifying the function g to $x^2 + y^2$ or $-x^2 - y^2$ in case (B) and to xy in (C). Such transformation (7) preserves (2), and one obtains normal forms (B) and (C) in §4.

Now suppose that condition (6) is violated. Since $j^2 g \neq 0$, we can assume that $A(0) \neq 0$, and further that $A(z, u, v) \equiv 1$. Using again the Morse lemma with parameters (the parameters are now y, z, u, v) we can claim the existence of a coordinate change of form (7) bringing the function g to the form

$$g = x^2 + y^2\tau(z, u, v) + y^3\nu(y, z, u, v), \qquad \tau(0) = 0. \tag{8}$$

For a generic differential system $d\tau(0) \neq 0$ (the condition $d\tau(0) = 0$ together with violating (6) and the inclusion of the source point 0 in $M_{\deg}(V)$ give the degeneration of codimension 6, nontypical by Transversality theorem). This condition is a bit stronger than the following one: the set of points $p \in M_{\deg}(V)$ such that the germ at p of a pair $(M_{\text{sing}}(V), M_{\deg}(V))$ is not equivalent either to normal form (B) or (C) in §4 is a smooth 2-dimensional submanifold. Therefore there exists a transformation of the form

$$z \to \eta_1(z, u, v), \quad u \to \eta_2(z, u, v), \quad v \to \eta_3(z, u, v) \tag{9}$$

simplifying expression (8) of g to

$$g = x^2 - y^2 z + y^3\tilde{\nu}(y, z, u, v). \tag{10}$$

A transformation of the form

$$z \to z + y\alpha(y, z, u, v)$$

reduces $\tilde{\nu}$ in (10) to

$$\hat{\nu} = \tilde{\nu}(y, z + y\alpha, u, v) - \alpha.$$

By the implicit function theorem we can ensure $\hat{\nu} = 0$ by choosing a suitable function α. Thus we arrive at normal form (D) in §4.

Finally, it is clear that normalizing the equation for $M_{\text{sing}}(V)$ above we preserve the description of $V(0) = \text{span}(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ valid for system V in normal form (cf. Corollary 2). The proof of the Main Theorem is complete. □

12

The proof of the theorem on normal form will be split into several assertions in this section and the next.

Lemma 2. *If* $(d\omega)^2(0) \neq 0$ *then* ω *is reducible to* $x\,du + y\,dv + \tau$, *where* $\tau \in \mathfrak{m}^2_{x,y}$.

Proof. Let us choose such coordinates that (2) holds. Then obviously $j^0_{x,y}\omega = 0$ and one has an expansion

$$j^1_{x,y}\omega = (xA_{11} + yA_{12})\,dx + (xA_{21} + yA_{22})\,dy + (xA_{31} + yA_{32})\,dz + (xA_{41} + yA_{42})\,du + (xA_{51} + yA_{52})\,dv, \tag{11}$$

where $A_{ij} = A_{ij}(z, u, v)$. Now we consider two 1-forms on $\mathbb{R}^3(z, u, v)$:

$$\mu_i := A_{3i}(z, u, v)\,dz + A_{4i}(z, u, v)\,du + A_{5i}(z, u, v)\,dv, \qquad i = 1, 2.$$

The transformation

$$\begin{aligned} x &\to \alpha_{11}(z, u, v) \cdot x + \alpha_{12}(z, u, v) \cdot y, \\ y &\to \alpha_{21}(z, u, v) \cdot x + \alpha_{22}(z, u, v)\, y \end{aligned} \tag{12}$$

brings $j^1_{x,y}\omega$ to the form

$$(\widetilde{A}_{11}x + \widetilde{A}_{12}y)\,dx + (\widetilde{A}_{21}x + \widetilde{A}_{22}y)\,dy + x\,\tilde{\mu}_1 + y\,\tilde{\mu}_2,$$

where

$$\tilde{\mu}_1 = \alpha_{11}\mu_1 + \alpha_{12}\mu_1, \qquad \tilde{\mu}_2 = \alpha_{21}\mu_1 + \alpha_{22}\mu_2.$$

Direct calculation shows that the assumption $(d\omega)^2(0) \neq 0$ means exactly $\mu_1 \wedge \mu_2|_0 \neq 0$ (the reason we have introduced μ_1, μ_2!). Apply any transformation (9) that straightens in $\mathbb{R}^3$ the line field $\ker \mu_1 \wedge \mu_2$ to $\text{span}(\partial/\partial z)$. Understanding it as the coordinate change in $\mathbb{R}^5$, one has then $\text{span}(\mu_1, \mu_2) = \text{span}(du, dv)$ in a neighbourhood of $0 \in \mathbb{R}^5$. This can be improved, using (12), to $\tilde{\mu}_1 = du$, $\tilde{\mu}_2 = dv$, yielding

$$j^1_{x,y}\omega = (x\widetilde{A}_{11} + y\widetilde{A}_{12})\,dx + (x\widetilde{A}_{21} + y\widetilde{A}_{22})\,dy + x\,du + y\,dv.$$

The change of coordinates

$$u \to u - x\widetilde{A}_{11} - y\widetilde{A}_{21}, \qquad v \to v - x\widetilde{A}_{12} - y\widetilde{A}_{22}$$

eventually annihilates $j^1_{x,y}(\omega - x\,du - y\,dv)$. □

Lemma 2 will serve as the premise for $k = 1$ in the inductive argument justifying Corollary 3 (see below). The following will constitute the induction step.

Lemma 3. *For any* $k \geq 1$ *and* ω *satisfying* $(d\omega)^2(0) \neq 0$ *the jet* $j^k_{x,y}\omega$ *is reducible to form* (1).

Proof. Suppose that for certain $k \geq 1$ $j^k_{x,y}\omega$ is already reduced to form (1). We shall show that $j^{k+1}_{x,y}\omega$ can be so reduced, too. Let $\mathfrak{m}^{(l)}_{x,y}$ be the set of all function germs of the form $\sum_{i+j=l} c_{ij}(z, u, v)x^i y^j$, where c_{ij} are germs at $0 \in \mathbb{R}^3$ of smooth functions. Write

$$(13) \qquad j^{k+1}_{x,y}\omega = j^k_{x,y}\omega + A_1\,dx + A_2\,dy + A_3\,dz + A_4\,du + A_5\,dv,$$

where $A_i \in \mathfrak{m}^{(k+1)}_{x,y}$. We are going to take new coordinates

$$(14) \qquad x+\varphi,\ y+\psi,\ z,\ u+\gamma,\ v+\mu,\ \text{where } \varphi, \psi, \gamma, \mu \in \mathfrak{m}^{(k+1)}_{x,y}.$$

Denoting by T the right-hand side of (13), $j^{k+1}_{x,y}\omega$ assumes in these coordinates the form $T + (x\gamma_x + y\mu_x)\,dx + (x\gamma_y + y\mu_y)\,dy + \varphi\,du + \psi\,dv$. So it suffices to solve the system

$$(15) \quad \varphi + A_3 = \psi + A_4 = 0, \quad x\gamma_x + y\mu_x + A_1 = 0, \quad x\gamma_y + y\mu_y + A_2 = 0.$$

Obviously $\varphi = -A_3$, $\psi = -A_4$. By putting $R := x\gamma + y\mu$ we reduce (15) to the system of three equations for R, γ, μ:

$$R_x - \gamma + A_1 = 0, \quad R_y - \mu + A_2 = 0, \quad R = x\gamma + y\mu,$$

$\gamma, \mu \in \mathfrak{m}^{(k+1)}_{x,y}$. This can be written briefly as

$$(16) \qquad R = x(R_x + A_1) + y(R_y + A_2), \qquad R \in \mathfrak{m}^{(k+2)}_{x,y}$$

(having such R, we take $\gamma := R_x + A_1$, $\mu := R_y + A_2$). As $xA_1 + yA_2 = \sum_{i+j=k+2} b_{ij}(z, u, v)x^i y^j$, we can give an explicit solution to (16):

$$R = \sum_{i+j=k+2} (1-i-j)^{-1} b_{ij}(z, u, v)x^i y^j. \quad \square$$

We denote by $\mathfrak{m}^{\infty}_{x,y}$ the ideal of germs of functions vanishing on $\{x = y = 0\}$ together with all their partial derivatives, and also the set of 1-form germs having such coefficients, and the set of respective vector field germs, too. Consequently, $j^{\infty}_{x,y}$ is defined analogously to $j^k_{x,y}$ (see §5).

Corollary 3. ω *as in Lemma* 3 *is reducible to* $x\,du + y\,dv + f\,dz + \tau$, *where* τ *is a* 1*-form,* $\tau \in \mathfrak{m}^{\infty}_{x,y}$, *and* $f \in \mathfrak{m}^2_{x,y}$.

Proof. Using Lemma 2 as the departure point ($k = 1$) we reduce inductively consecutive jets $j^k_{x,y}\omega$, applying Lemma 3 at each step. Taking into account the character of normalizing transformations (14) and the fact that, after passing to a fixed representative of ω, (16) is solvable in an independent of k neighbourhood of $0 \in \mathbb{R}^3(z, u, v)$, there exists a formal in x, y transformation of $\mathbb{R}^5$, having as coefficients (of its series in x, y) smooth functions of z, u, v defined in a common neighbourhood of 0, which reduces $j^{\infty}_{x,y}\omega$ to form (1). Now it suffices to apply the Whitney extension theorem (see [W]). $\square$

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To prove the theorem on normal form we must still prove

Lemma 4. *Let* $f \in \mathfrak{m}^2_{x,y}$, τ *be a* 1*-form,* $\tau \in \mathfrak{m}^\infty_{x,y}$. *Then the* 1*-form* $\omega = x\,du + y\,dv + f\,dz + \tau$ *is reducible to the form* $x\,du + y\,dv + \tilde{f}\,dz$, $\tilde{f} \in \mathfrak{m}^2_{x,y}$.

Proof. We use some modifications of the homotopy method [Z2, Chapter 1, §3]. Let $\hat{\omega} := x\,du + y\,dv$. Introduce also the truncation operator P sending every 1-form $\kappa_1\,dx + \kappa_2\,dy + \kappa_3\,dz + \kappa_4\,du + \kappa_5\,dv$ (κ_i are functions of x, y, z, u, v) to $\kappa_1\,dx + \kappa_2\,dy + \kappa_4\,du + \kappa_5\,dv$, and the family of forms $\omega_t := \hat{\omega} + t(f\,dz + \tau)$, $t \in [0, 1]$. Consider the equation

$$P(X_t \lrcorner\, d\omega_t + d(X_t \lrcorner\, \omega_t) + f\,dz + \tau) = 0 \tag{17}$$

for an unknown family of vector fields X_t. (We shall require additionally that $X_t \lrcorner\, dz \equiv 0$.) Equation (17) can be equivalently written as

$$P(X_t \lrcorner\, d\omega_t + d(X_t \lrcorner\, \omega_t) + \tau) = 0. \tag{18}$$

Claim. If there exists a smooth family $X_t \in \mathfrak{m}^\infty_{x,y}$ depending on t and satisfying (18) and such that $X_t \lrcorner\, dz \equiv 0$, then Lemma 4 holds. (Compare the classical variant of the homotopy method [AGV].) In order to substantiate the Claim, consider the family of diffeomorphisms ϕ_t defined by

$$\frac{d\phi_t}{dt} = X_t(\phi_t), \qquad \phi_0 = \mathrm{id}\,.$$

Then

$$\begin{aligned}\frac{d}{dt}(P(\phi_t^*\omega_t)) &= P\left(\frac{d}{dt}(\phi_t^*\omega_t)\right) = P\phi_t^*\left(L_{X_t}\omega_t + \frac{d\omega_t}{dt}\right)\\ &= P\phi_t^*(X_t \lrcorner\, d\omega_t + d(X_t \lrcorner\, \omega_t) + f\,dz + \tau),\end{aligned}$$

where $L_{X_t}\omega_t$ is the Lie derivative of ω_t along the field X_t. Equation (17) implies that $L_{X_t}\omega_t + d\omega_t/dt \in \ker P \;\; \forall t$. By virtue of $X_t \lrcorner\, dz \equiv 0$, also $\phi_t^*(L_{X_t}\omega_t + d\omega_t/dt)$ is always included in $\ker P$; hence $d(P(\phi_t^*\omega_t))/dt \equiv 0$. This infers $P\phi_1^*\omega_1 = P\phi_0^*\omega_0$, or $\phi_1^*\omega - \hat{\omega} \in \ker P$, i.e., $\phi_1^*\omega - \hat{\omega} = \tilde{f}\,dz$. Noticing that $\phi_t = \mathrm{id} + \varphi_t$, $\varphi_t \in \mathfrak{m}^\infty_{x,y}$, the assumption $f \in \mathfrak{m}^2_{x,y}$ obviously implies $\tilde{f} \in \mathfrak{m}^2_{x,y}$, proving the Claim.

Now we are going to show that the premise in the Claim holds. Seeking X_t in the form $f_{1,t}\partial/\partial x + f_{2,t}\partial/\partial y + f_{3,t}\partial/\partial u + f_{4,t}\partial/\partial v$, $f_{i,t} \in \mathfrak{m}^\infty_{x,y}$, (18) boils down to a system of five equations for the $f_{i,t}$'s and $R_t := X_t \lrcorner\, \omega_t$, the last defining identity being itself the fifth equation which, on writing $\tau = \tau_1\,dx + \tau_2\,dy + \tau_3\,du + \tau_4\,dv$ and computing $X_t \lrcorner\, \omega_t$ explicitly, assumes the form

$$R_t = t\tau_1 f_{1,t} + t\tau_2 f_{2,t} + (x + t\tau_3)f_{3,t} + (y + t\tau_4)f_{4,t}\,.$$

The unknowns $f_{i,t}$ can be eliminated from the first four equations (they can be expressed via the first order partial derivatives of R_t), after which we arrive at one equation for R_t:

$$R_t - x(R_t)_x - y(R_t)_y + \Theta_t(R_t) = a_t\,, \tag{19}$$

where Θ_t is a family of vector fields, a_t is a family of function germs, $\Theta_t \in \mathfrak{m}^\infty_{x,y}$, $a_t \in \mathfrak{m}^\infty_{x,y}$. (Observe that (19) is, to some extent, similar to (16), but the occurring flat function and vector field make the great difference.)

The fields $-x\partial/\partial x - y\partial/\partial y + \Theta_t$ are hyperbolic on the manifold $\{x = y = 0\}$ (this manifold is attracting for the respective dynamical system). Thanks to that, by virtue of the Belitskii results (see [B1, B2] and references in the latter[1]), (19) has a smooth family of solutions $R_t \in \mathfrak{m}^{\infty}_{x,y}$ depending on t. The proof of Lemma 4 is finished. □

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[1]Various results by Belitskii on the solvability of singular partial differential equations can be found in [Z2, Chapter 1, §6], where they are collected in a compact form.